Title: Developing Mathematical Logic Skills

Brief Overview:

This series of lessons will introduce fundamental logic concepts including recognizing and constructing statements, Boolean logic, forming conditional statements and their various constructs, forming valid arguments, and drawing valid conclusions.

Links to NCTM Standards:

- **Mathematics as Problem Solving**
  Logic skills will be used to organize components of a problem. By recognizing relevant information and valid arguments, ordering conclusions sequentially and ruling out possibilities, the problem solving process will be streamlined.

- **Mathematics as Communication**
  Mathematical symbols will be used to help construct statements and arguments. This will also enable the students to consider the more rigorous expression of every-day verbal statements. Verbal and mathematical statements will be written using standard mathematically logical constructs.

- **Mathematics as Reasoning**
  The basis for this learning unit is to help students reason more rigorously; a fundamental skill to all mathematics.

- **Mathematical Connections**
  Logic and reasoning skills developed through this learning unit will enable students to write and speak more effectively. Arguments and positions posed in other subject areas can be more thoroughly examined. Arguments and positions claimed will be better supported by applied skills from this unit.

- **Geometry**
  Knowledge of the definition of specific geometric forms will be applied in the final activity.

Grade/Level:

This learning unit is designed for grades 6-8.

Duration/Length:

An estimated 5 hours (in five one hour lessons) would be required for all of the activities.

Prerequisite Knowledge:

Students should have working knowledge of the following skills:

- Basic set theory especially how to draw Venn diagrams
- Definitions of quadrilateral, square, rectangle and polygon

Objectives:

Students will:

- be able to recognize and form basic statements and their negations.
- understand the Boolean concepts of "and" and "or," and express them using Venn diagrams.
- be able to recognize and form Boolean statements and their negations, and to construct truth tables.
- be able to recognize and form a conditional statement as well as the negation, the inverse, the converse and the contra positive of said statement.
- use mathematical symbols to express statements.
- recognize and form valid arguments given certain conditions.
- draw valid conclusions from given conditions.

Materials/Resources/Printed Materials:

- Chalkboard and chalk
- Index cards (for final activity)

Development/Procedures:

Lesson 1 - Making a Statement:

Teacher defines statements as sentences that can be determined as either true or false. Make sure that students understand that statements must be objectively true or false, and that subjectivity is not accepted. Students are then provided with examples of simple statements. They are to determine if the sentence is a statement. If it is a statement, students are then to determine if it is true or false. Some examples might be as follows:

1.	People are mammals.	*(A true statement.)*
2.	2 + 4 = 5	*(A false statement.)*
3.	The grass is green.	(A true statement.)
4.	She is 10 years old.	*(Not a statement. We don't know who "she" is.)*
5.	x + 3 = 7	*(Not a statement. We don't know what "x" is.)*
6.	My mother is pretty.	*(Not a statement. Pretty is subjective.)*

Introduce the universal quantifiers "each," "every," and "for all" as well as the existential quantifier "there exists." Have the students reword examples that are not statements using both a universal and existential quantifier. Then have them determine if the new sentence is a statement and, if so, whether the statement is true or false.

Using the preceding examples, the results would be as follows:

4.	All girls are 10 years old.	*(A false statement.)*
	There is a girl who is 10 years old.	*(A true statement.)*
5.	For all x, x + 3 = 7.	*(A false statement.)*
	There is an x where x + 3 = 7.	*(A true statement.)*

Introduce the symbols for the universal and existential quantifiers and show the students how to use them. Use an example like #5 in the preceding examples to demonstrate their usage.

5. $\forall$ x: x + 3 = 7 *For all x, x + 3 equals 7.*
   $\exists$x $\exists$x + 3 = 7 *There exists an x such that x + 3 = 7.*

Have students negate previous statements (including the sentences that were later made into statements using quantifiers. To negate quantified statements, change the quantifier from universal to existential, or vice versa, and negate the part after the quantifier. Using the previous example and paying particular example to numbers 4 and 5, the results would be as follows.

1. People are not mammals.
2. $2 + 4 \neq 5$
3. The grass is not green.
4. $\forall$x $\exists$x + 3 $\neq$ 7.
5. $\exists$ x, x + 3 $\neq$ 7.

Finally, have the students construct ten (or more) statements and negate them. They should use logical quantifier symbols whenever possible.

Lesson 2 - An Introduction to Boolean Logic:

Explain the difference between “and” and “or”. Show how in order for an “and” statement to be true, both parts of the statement must be true. For an “or” statement to be true either one or the other or both must be true.

Present examples of statements that combine two (or more) simple statements. Use Boolean operators. Ask students to determine if the statements are true or false. Some examples might be as follows:

1. The sky is blue and the ocean is blue.
   In order to be true, the sky and the ocean both would have to be blue. Generally, this is a true statement.

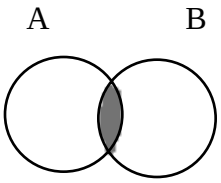
2. You can read “The Little Prince” in English or in French.
   The book “The Little Prince” is translated into both French and English. Therefore you can read “The Little Prince” in French or English or both. The most accurate understanding of this “or” is “and/or.” This is a true statement.

Then have the students negate the statements.

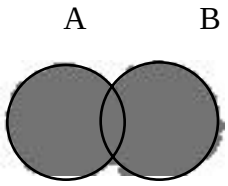
1. The sky is not blue or the ocean is not blue.
   In order to negate an “and” statement you make it an “or” statement and negate both parts of the statement.

2. You cannot read “The Little Prince” in English and you cannot read it in French.
   In order to negate an “or” statement, you must make it the equivalent of an “and” statement and negate both parts of the statement.

Use Venn diagrams to show how "and" and "or" are related to union and intersection.



A and B (A ∩ B)



A or B (A ∪ B)

Have students write their own compound statements and use Venn diagrams (whenever possible) to illustrate structure of the statements. Then have them write the negation.

Lesson 3 - Conditionals:

Show students the standard format for an "if-then" statement.

For all p, if p then q.

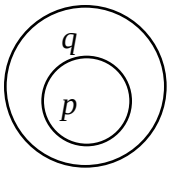
Show students how to construct statements in "if-then" form given a conditional statement. Students should underline the "p" and "q" parts of the sentence. They should also state whether the statement is true or false. Use examples similar to the following:

1. It snows only if the sky is cloudy.
   If it snows, then the sky is cloudy. (true)

2. All politicians are dishonest.
   If someone is a politician, then (s)he is dishonest. (false)

3. No even numbers are prime.
   If a number is even, then the number is not prime. (false-2 is the exception)

Use Venn diagrams to show the relationship between "p" and "q".



If p then q requires that for all instances (or elements) which satisfy p, q must be true also. Therefore, the set of elements satisfying p is a subset of the set of all elements satisfying q.

Using the same examples, demonstrate how to negate conditional statements. Note that negating a conditional statement is not as simple as negating one part of the statement or the other. The negation of "∀x, if p then q" is "∃x so that p is true and q is false".

1. It snows only if the sky is cloudy.
   Sometimes it snows when it is not cloudy.
   or
   There exists a time when it snows that the sky is not cloudy.

2. All politicians are dishonest.
   Some politicians are not dishonest.
   or
   There exists a politician who is not dishonest.

3. No even numbers are prime.
   There is an even number that is prime.

Have the students write a series of conditional statements in "if-then" form and write the negation of each statement.

Lesson 4 - Forming a Valid Argument

Given a conditional statement as a premise, show students how to develop a valid argument using the following methods:

Direct	**Example**
If p then q	If it is raining then it is cloudy.
p	It's raining.
□ q	Therefore, it's cloudy.

Contra positive	**Example**
If p then q	If it is raining then it is cloudy.
not q	It's not cloudy.
□ not p	Therefore, it's not raining.

Transitivity	**Example**
If p then q	If it is raining then it is cloudy.
If q then r	If it's cloudy then it's dark.
□ If p then r	If it's raining then it is dark.

Ruling Out Possibilities	**Example**
p or q	Either it is cloudy or it is sunny.
not p	It's not cloudy.
□ q	Therefore, it's sunny.

Contradiction	**Example**
p	It's raining.
q or r	Either it is sunny or it is cloudy.
If q then not p	If it is sunny then it is not raining.
□ r	Therefore, it's cloudy.

Demonstrate that the following are not valid arguments. Use examples so that the students can see why they are not.

Converse	**Example**
If p then q ≠> if q then p.	If it is raining then it is cloudy *does not imply* if it is cloudy then it is raining.
Inverse	**Example**
If p then q ≠> if not p then not q.	If it is raining then it is cloudy *does not imply* if it is not raining then it is not cloudy.

Have the students write their own conditional statements and derive valid arguments or conclusions from the statements.

Lesson 5 - The Burden of Proof

This is a simulated court case of a quadrilateral (maybe a rhombus) accused of being a square. Insufficient "evidence" is provided to conclude that the quadrilateral is a square. Students are to be presented with opening statements from a defense attorney and a prosecutor followed by eye witness testimonies. From the statements and the testimony, students in the jury are to consider the evidence and draw valid conclusions. Students must be able to explain how they arrive at their conclusion. If they do not arrive at any conclusion, they must explain why not.

For this exercise, you will need four (4) copies of the following script and several index cards. This exercise will use mathematical concepts to demonstrate how valid arguments are (or should be) used in a courtroom.

- Assign four students the following roles:

  Judge
  Defendant
  Defense Attorney
  Prosecuting Attorney

  These students will receive a short script.

- Select six students to be witnesses. Each student will receive an index card with a statement written on it.

- The remaining students will be on the jury. Jurists are instructed to take thorough notes on testimonies presented.

- Have students read their lines from the following script.

Judge: Court will now come to order. (To the defendant). You are accused of being a square. How do you plead?

Defendant: Not guilty.

Judge: We will now hear from the defense.

Defense: People of the court, my client is accused of being a square. The burden of proof is on the prosecution. He is innocent until proven guilty. I contend that the prosecution does not have sufficient evidence to convict my client.

Judge: Prosecution. Your opening statement please.

Prosecution: We have sufficient evidence and numerous witnesses do conclude that the defendant is clearly guilty. (Prosecution proceeds to call each witness one at a time).

The following are witness statements. They should be written on index cards and distributed to the students selected to be witnesses. Witnesses may testify in any order. Additional "testimonies" can be written by the teacher to engage more students. However, the testimonies should not alter the logical conclusions drawn during the case.

Witness: At club triangle only three sided polygons are allowed. The defendant simply had too many sides. I had to turn him away.

Witness: I work in human resources at the Pentagon. In order to work at the Pentagon you must have at least five sides. The defendant was under qualified. I didn't even consider his application.

Witness: I am a bouncer at the Polygon Club. To get in you must be a polygon with three or more sides. No curves allowed. The defendant comes here all the time.

Witness: I'm an expert on rectangles. All rectangles have four sides. All squares are rectangles with equal sides and equal angles.

Witness: I'm an equilateral triangle. I like talking to the defendant. He understands me, both of us having equal sides and all.

Witness: The defendant is my neighbor. He's as nice as can be, but I don't think he's quite right.

Judge: At this time I will dismiss the jury for deliberation. Please come back in a few minutes with your verdict. Remember it must be a unanimous decision to convict the defendant.

At this time the jurors deliberate. They should write testimonies as conditional statements and consider which statements are relevant to the case. Then based on the information provided, they should draw a logical conclusion.

Performance Assessment:

Teachers can assess students' progress throughout the series of lessons by evaluating each students work at the end of each lesson. For lessons one through four, students should be given a set number of examples to create and manipulate. Each example should be worth a set number of points that considers each manipulation. Teachers can also develop and distribute their own examples for students to identify the parts of the statements, conclusions, etc.

For example, in lesson four, students could come up with five conditional statements and be required to construct a direct argument and an argument based on the contra positive. Each example would qualify for 5 points. 1 point for a correctly constructed conditional statements. 2 points each for correctly constructed arguments. The total points possible would be 25 points.

Extension/Follow Up:

There are a myriad of logic puzzles available in various books and on the Internet. It would be a great exercise to have the students find their own and use them in the classroom. They could present their solutions to the class, explaining how they used valid argument forms, such as direct argument and ruling out possibilities. Another extension would be for the students to find newspaper or magazine articles that attempt to argue a point and analyze their logical arguments. They can identify the form of argument used and determine whether it is valid.

Authors:

Brooke Abercrombie
Roadnet Technologies
Baltimore/Howard Counties

Geoffrey Birky
Greenwood Mennonite School
Sussex County, DE

Dan Atkinson
NSA
Fort Meade, MD